MAT 534 FALL 2015 REVIEW FOR MIDTERM I

General

The exam will be in class on Thursday, October 1. It will consist of 5 problems and will be a closed book exam on group theory, covering section 1)-10) below.

Material covered in class

1) Basic group theory: groups, homomorphisms, subgroups, cyclic groups. For any $S \subseteq G$ the normalizer $N_G(S)$ and the centralizer $C_G(S)$ of S in G are subgroups of G and $Z(G) = C_G(G)$ is the center of G. Cosets and quotient spaces, Lagrange's Theorem: if G is a finite group and $H \leq G$, then
$$|G| = |G : H||H|, \quad \text{where} \quad |G : H| = |G/H|.$$
2) Normal subgroups, $N \trianglelefteq G$ iff $N_G(N) = G$, quotient groups G/N and canonical homomorphism
$$\pi_N : G \to G/N, \quad \ker \pi_N = N.$$
The isomorphism theorems.
3) Direct and semi-direct products. Isomorphism criterion for semi-direct products: $N \rtimes_{\varphi_1} K \cong N \rtimes_{\varphi_2} K$ if there is $f \in \mathrm{Aut}(K)$ such that $\varphi_2 = \varphi_1 \circ f$. The recognition theorem: if $N, K \trianglelefteq G$ and $N \cap K = \{1\}$, then $HK \cong H \times K$.
4) Symmetric and alternating groups.
5) Definition and examples of simple, solvable and nilpotent groups.
6) Group actions, orbits and stabilizers. Orbit decomposition
$$A = \bigcup_{i \in I} \mathcal{O}_{a_i}$$
— disjoint union, where $\mathcal{O}_{a_i}$ are orbits of a G-action on A. The counting formula for finite group G:
$$|\mathcal{O}_a| = |G : G_a|,$$
where $\mathcal{O}_a = G \cdot a$ is the orbit of $a \in A$ and G_a is the stabilizer of $a \in A$. Orbit decomposition formula: if $|A| < \infty$,
$$|A| = \sum_{i \in I} |G : G_{a_i}|.$$

7) Action of G on G by left multiplication (the left regular action), action of G on cosets G/H and on the subsets of G.
8) Action of G on G by conjugation, orbit $\mathcal{O}_g$ is the set of conjugacy classes of $g \in G$, $G_g = C_G(g)$ (the case when $S = \{g\}$). Action of G by conjugations on the set of subsets of $S \subseteq G$, $G_S = N_G(S)$, normal subgroups as the fixed points of this action. The class equation
$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|,$$
where $g_1, \ldots, g_r$ are representatives of all conjugacy classes in G that are not in $Z(G)$.
9) Solvability of p-groups. Fixed point theorem for p-groups: if a p-group P acts on A and $(p, |A|) = 1$, then there is a fixed point.
10) Sylow's theorems and classification of groups of small orders, of order pq with $p < q$, basic examples.
11) Free and torsion abelian groups, rank. The elementary divisor decomposition of a finite abelian group and the primary divisor decomposition of a finite abelian p-group. The main theorem for finitely generated abelian groups.
12) Free groups with n generators, free products of groups.